gas

Gas Dynamics :

is the fluid dynamics of compressible flows and de with the unified analysis of dynamics and thermodynami of such flows. The analysis of such high speed flows o gases and vapours are inadequate without considering compressibility that will produce in a fluid by a specific change in pressure. In a fluid flow, there are usually changes in pressure associated with changes in the velocity in the flow. These pressure changes will in gener introduce density changes which will have an influence the flow. If the density changes are important, the cha in temperature in the flow that arise due to kinetic ener changes associated with the important velocity changes al influence the flow. All fluid (gases, vapours) compre if the pressure increase resulting in a decrease in volume.

Coefficient of compressibility $(B) = \dfrac{\text{relative change in Volu}}{\text{Change in Pressure}}$

$$B = \lim_{\Delta p \to 0} -\frac{\Delta V/V}{\Delta p} = -\frac{1}{V}\frac{\partial V}{\partial p} = -\frac{1}{v}\frac{\partial v}{\partial p}$$

v is specific volume $= \dfrac{1}{\rho}$,

$$\therefore B = -\frac{1}{1/\rho}\frac{\partial(1/\rho)}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \quad m^2/N \qquad (1.1)$$

$v = \frac{1}{\rho} = \partial v = -\frac{1}{\rho^2} d\rho$

The bulk Modulus E

The bulk modulus of elasticity E = Increase in Pressure / Relative change in pressure

$$E = \rho \frac{\partial p}{\partial \rho} \quad N/m^2 \; (Pa) \quad (1.2)$$

$\frac{d\rho}{\rho^2}$ $\frac{1}{d\rho}$

Fundamental Assumptions

(a) Gas is continuous (b) No chemical changes

(c) Gas is perfect

(d) The specific heats are constants,

$$C_v = \left(\frac{\partial u}{\partial T}\right)_v \quad \text{and} \quad C_p = \left(\frac{\partial h}{\partial T}\right)_p \qquad (2)$$

$$k = \text{Specific heat ratio} = \frac{C_p}{C_v} \qquad (3)$$

$$R = C_p - C_v = \text{gas constant} \quad J/kg\,K \qquad (4)$$

$$R = \frac{R_o}{M} \qquad (5)$$

where $R_o = 8314 \; J/kgmol\,K$.

M = molecular mass $kg/kgmol$

(e) Gravitational effects on the flow are negligible.

(f) Magnatic and electrical effects are negligible.

(g) The effects of viscosity are negligible.

(h) Steady state

All compressible fluid flow equations are derived from

(1) Conservation of mass (continuity equation)

(2) Conservation of momentum (Newton's 2nd law)

(3) Conservation of energy (1st law of thermodynamics)

(4) Equation of state.

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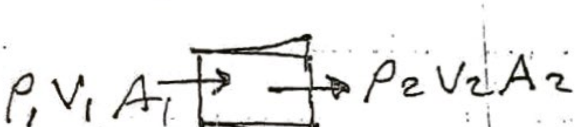
1 Conservation of Mass (1)

Rate of increase mass in C.V. = (Rate of mass)$_{in}$ − (Rate of mass)$_{out}$

Compressible steady flow

$\dot{m}$: kg/s



$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \text{constant}$$

$$\dot{m} = \rho V A = \text{constant} \qquad (6)$$

(d) The specific heats are constants,

$$C_v = \left(\frac{\partial u}{\partial T}\right)_v \quad \text{and} \quad C_p = \left(\frac{\partial h}{\partial T}\right)_p \qquad (2)$$

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قانون حفظ الكتلة (1)

1. Conservation of Mass

Rate of increase mass in C.V. = (Rate of mass)$_{in}$ − (Rate of mass)$_{out}$

Compressible steady flow $\qquad \dot{m} = 0 \quad kg/s$

سائل $\rho_1 V_1 A_1$  $\rho_2 V_2 A_2$

$$\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \text{constant}$$

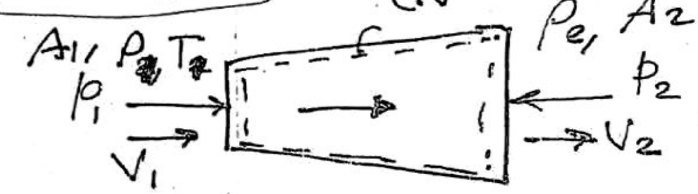
$$\dot{m} = \rho V A = \text{constant} \qquad (6)$$

قانون حفظ زخم

2 Conservation of Momentum (Newton 2nd Law)

Net force on gas in control volume in direction considered = Rate of momentum leaves the C.V. in direction considered − Rate of momentum enters C.V. in direction considered

$$\boxed{\sum \vec{F} = m\frac{dV}{dt}} = (\dot{m}V)_{out} - (\dot{m}V)_{in}$$



of momentum: sum of force in direction of motion

$\dot{m} = \rho A V$,

$$P_1 A_1 - P_2 A_2 = \dot{m}_2 V_2 - \dot{m} V_1 = A_2 \rho_2 V_2^2 - A_1 \rho_1 V_1^2 \quad (7)$$

3 First Law of Thermodynamics or Conservation of Energy

$\sum f = m a$

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For open system;

$$Q = W + \Delta U + \Delta KE + \Delta PE = W + \Delta E$$

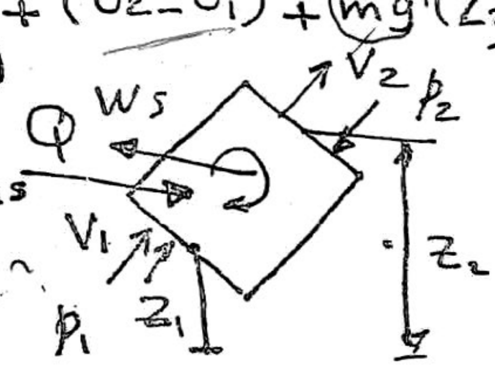
$$W = W_s + (P_2 \forall_2 - P_1 \forall_1)$$

= Shaft Work + Work done on the Boundary

shaft work

$$Q = W_s + (P_2 \forall_2 - P_1 \forall_1) + (U_2 - U_1) + mg(Z_2 - Z_1) + \frac{1}{2} m (V_2^2 - V_1^2)$$

????



From the previous assumptions (?)

$Q = 0$, $W_s = 0$

neglecting gravitational force $\Delta PE = 0$, and we end with

$$(P_2 \forall_2 + U_2) + \frac{1}{2} m V_2^2 = (P_2 \forall_1 + U_1) + \frac{m}{2} V_1^2$$

W) ΔU ΔKE

$h = U + PV$

$P_1 V_2$???

isentropic = adiabatic + re

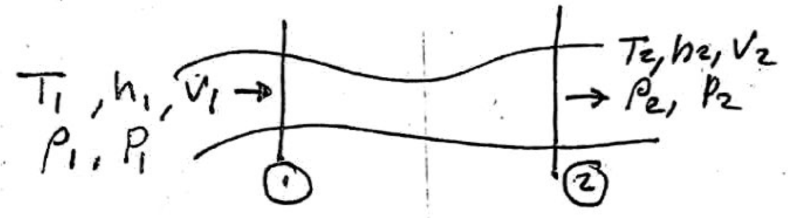
reversible

so, per unit mass, $(p_2 v_2 + u_2) + \frac{V_2^2}{2} = (p_1 v_1 + u_1) + \frac{V_1^2}{2}$

where $pv + u = h$ = specific enthalpy

so, $h_2 + \frac{V_2^2}{2} = h_1 + \frac{V_1^2}{2}$ = constant (7.1)

and we can apply it along the flow,



h_1, h_2 are the local enthalpies

$\frac{V_2^2}{2}, \frac{V_1^2}{2}$ are the dynamic enthalpies

and their sum is called total enthalpy,

recall that $h = C_p T$ (Refrence the absolute zero temper.

$$C_p T_2 + \frac{V_2^2}{2} = C_p T_1 + \frac{V_1^2}{2} = h_0 = \boxed{C_p T_0}$$

$C_p T_1 + \frac{V_1^2}{2} = C_p T_0$

$$\boxed{T_0 = T_2 + \frac{V_2^2}{2C_p}} = \boxed{T_1 + \frac{V_1^2}{2C_p}} \quad (8)$$

dynamic velocity

Total Temperature = Local Temperature + Dynamic Temperature

4. Second law of Thermodynamics

$$ds \geq \int \frac{dQ}{T}$$

For irreversible process $ds > \oint \frac{\delta Q}{T}$ (9)

where $\oint \frac{\delta Q}{T} \leq 0$ (Clausius's inequality)

For reversible process $ds = S_2 - S_1 = 0$ (10)

$$\oint \frac{\delta Q}{T} = 0 \quad (11)$$

$T ds = \delta Q$

1st law states that

$$\delta q = \delta W + du \quad (12)$$

$\delta W = p\,dv$, $\delta q = T\,ds$, $du = C_v\,dT$

$dh = T.d$

$T\,ds = p\Delta V$ ... da M

... $h = pV + u$

- Equation (12) becomes $\boxed{T\,ds = p\,dv + du}$ (13)

Differentiating implicitly $h = pv + u$ specific Volume (14)

$dh = p\,dv + v\,dp + du$

From (13) and (14) $T\,ds = dh - v\,dp$ (15)

$T\,ds = dh - \frac{dp}{\rho}$

5. Equation of State

$pV = mRT$, $m = nM$, $R = \frac{R_o}{M}$ universal gas constant $8314 \frac{J}{kgmol\,K}$

where n is a number of moles

$pV = nR_oT$

$p\frac{V}{m} = pv = RT$ or $p = \rho RT$ (16)

Recall (15) and we know that $dh = C_p\,dT$

$\therefore T\,ds = C_p\,dT - v\,dp$

$ds = C_p\frac{dT}{T} - \frac{v}{T}\,dp$, From (16) molecular mass kg/kgmol

$\int_1^2 ds = \int_1^2 C_p\,dT/T - R\int_1^2 \frac{dp}{p}$

$s_2 - s_1 = C_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$ (17)

$C_p - C_v = R$ or $\frac{C_p}{C_v} - 1 = \frac{R}{C_v}$

$\therefore \boxed{C_v = \frac{R}{k-1}}$, $k = \frac{C_p}{C_v}$ and $\boxed{C_p = \frac{k}{k-1}R}$ (18)

$\frac{p}{\rho} = RT$ or $p = \rho RT$

Substitute the above relations in (17)

$\frac{s_2 - s_1}{R} = \frac{k}{k-1}\ln\frac{T_2}{T_1} - \ln\frac{p_2}{p_1}$

For isentropic (adiabatic + reversible), $s_2 - s_1 = 0$

$\ln\left(\frac{T_2}{T_1}\right)^{k/k-1} = \ln\frac{p_2}{p_1}$ $T\,ds = dh + \frac{dp}{\rho}$

So, $\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{k/k-1}$, $p_2v_2^k = \text{cons} = p_1v_1^k$

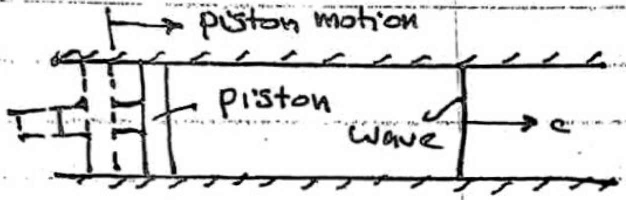
$\frac{p_2}{p_1} = \left(\frac{v_2}{v_1}\right)^k$ $\therefore \frac{v_1}{v_2} = \left(\frac{T_2}{T_1}\right)^{1/k-1} = \frac{\rho_2}{\rho_1}$ (19)

(5)

$R = m/J$

Speed of Sound

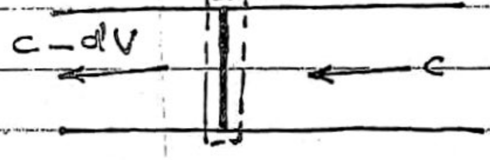
The speed of sound is the speed at which very weak pressure waves are transmitted through the gas. Consider a long duct with a piston shown in the figure below



A small movement of a piston generates a wave moving down a duct (infinitesmal pressure wave) with velocity c in a stagnant gas. dV is the piston velocity which

is imparted to the gas. Suppose that an observer moves with the wave. In this case the stagant gas at pressure p on the right appears to flow toward the left with a velocity c. When the flow has passed through the wave to the left its pressure is raised to p+dp and the velocity lowered to c−dV

A is the wave face area.



Applying the continuity equation;

$\dot{m} = \rho A c = (\rho + d\rho) A (c - dV)$; $\rho c = \rho c - \rho dV + c\, d\rho$

$\therefore \quad \rho\, dV = c\, d\rho$ ———— (20)

Applying momentum equation;

$A[p-(p+dp)] = \dot{m}\,[(c-dV)-c] = -\dot{m}\, dV$

$= -\rho A c\, dV$

$\therefore \boxed{dp = \rho\, c\, dV}$ ———— (21) ; $\dot{m} = \rho A c$

$\therefore$ From (20); $dV = \frac{c}{\rho} d\rho$ substitute in (21)

$dp = \rho c \cdot \frac{c}{\rho} d\rho$

$$c^2 = \frac{dp}{d\rho}$$

$$c = \left(\sqrt{\frac{dp}{d\rho}}\right)_s \quad (22) \quad m/s$$

sound velocity

From the defition of E equation (1)

$$\boxed{\frac{dp}{d\rho}} = \frac{E}{\rho}$$

C solid $> C$ liquid

$$\therefore c = \sqrt{\frac{E}{\rho}}$$, where E is the modulus of elastic and ρ the substance density

Consider a perfect gas with isentropic process

$$\frac{p}{\rho^k} = c \quad (23)$$

$p\rho^{-k} = c$

$$dp = k c \rho^{k-1} d\rho \Rightarrow \boxed{\frac{dp}{d\rho}} = \frac{k}{\rho} c \rho^{k}$$

From (23), $\frac{dp}{d\rho} = \frac{k}{\rho} \frac{p}{\rho^k} \cdot \rho^k = k \frac{p}{\rho}$

and $\frac{p}{\rho} = RT$

so, $\frac{dp}{d\rho} = kRT$

Gas dinamic isentropic process

$$(24) \quad \boxed{c = \sqrt{kRT}} \rightarrow (m/s)$$

Sound speed or velocity

Note: T must be absolute in K.

From the definition of $c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{E}{\rho}}$

the $c|_{solid} > c|_{liquid} > c|_{air\ or\ gas}$

Mach Number (M)

The Mach number of a moving object (aircraft or missile) is the ratio of its velocity and the velocity of sound in same medium.

So, $M = \frac{V}{c}$ (2.5)

4)

Mach number can also be obtained from

$$M^2 = \frac{\text{Inertia Force}}{\text{Elastic Force}} = \frac{\rho A\, V^2}{EA}$$

$\frac{m^{\circ} \cdot V}{EA}$

where $c = \sqrt{\frac{E}{\rho}}$ $\therefore E = \rho c^2$

$E = \rho c^2$

$$\therefore M^2 = \frac{\rho A V^2}{\rho A c^2} \Rightarrow \frac{V^2}{c^2}$$

$$\boxed{\therefore M = \frac{V}{c}}$$

Vlosity of object

Sound velocity in m,

The mach number is an index to classify ~~heat~~ / am the type of flow.

$0 < M < 0.3$	Incompressible flow	
$0.3 < M < 1$	subsonic flow	تحت الصوت
$M = 1$	Sonic flow	الحد الصوتي
$1 < M < 5$	Supersonic flow	فوق الصوت
$M > 5$	Hypersonic flow	اعلى

The applications of gas dynamics in earodynamics; space ~~vehicle~~ crafts, airplanes ... etc. and gas, steam turbines.

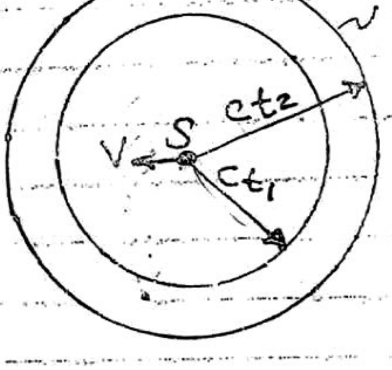
01

Type of Regions and Wave Motion

Suppose there is a movement of a source of disturbance (S) at a velocity V in a fluid from right to the left

(1) When the source is stagnant or moves at very low velocity V compare to sound velocity c, this produces infinitesimal spherical wave (pressure or sound waves move with velocity c.

object moves with low velocity. splecri کروية

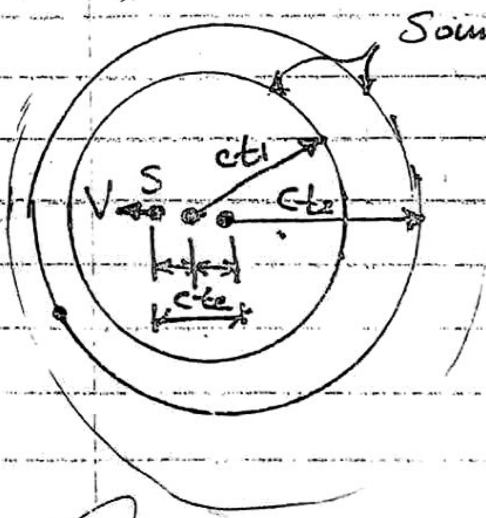


$V \approx 0$

$M = 0$

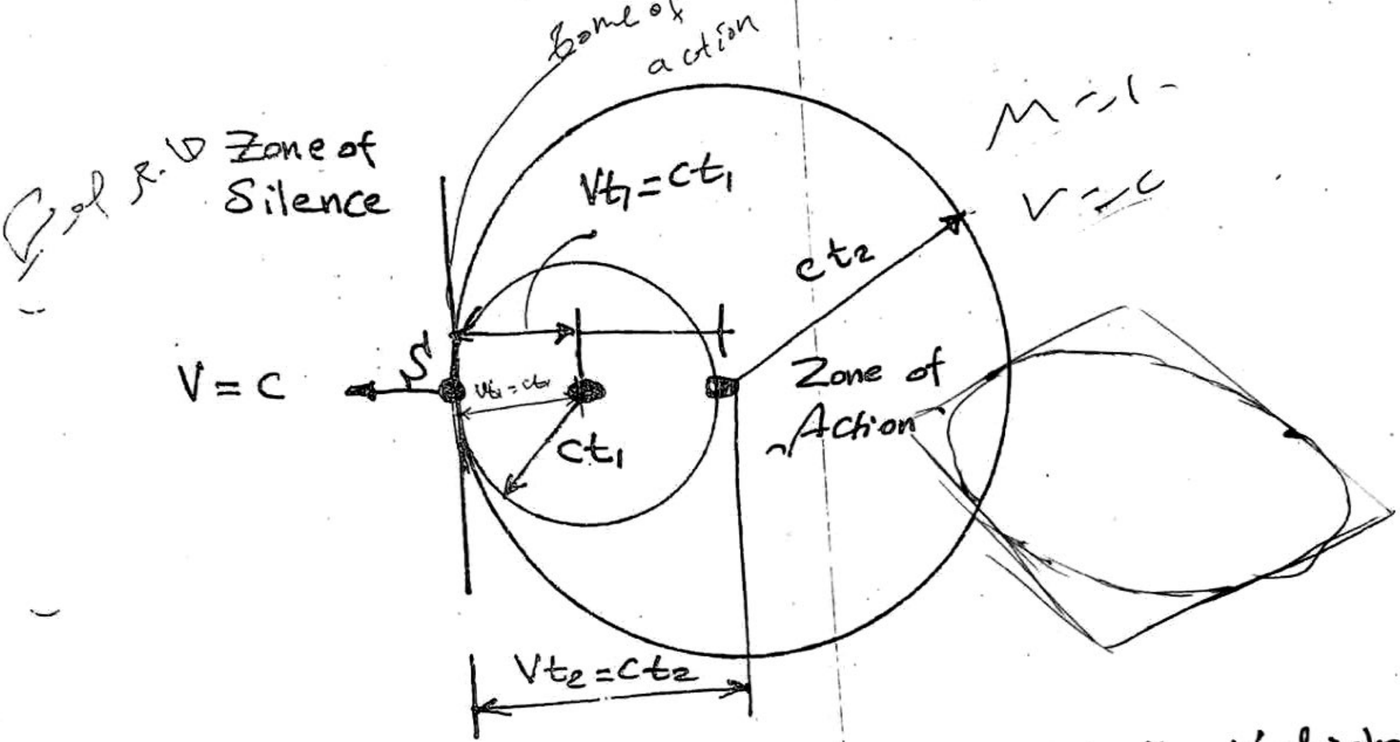
$c \neq 0$

(2) When the disturbance source (object) moves with V which is less than c, or $M < 1$. The flow is called subsonic, spherical sound waves generated and moved a head of S.

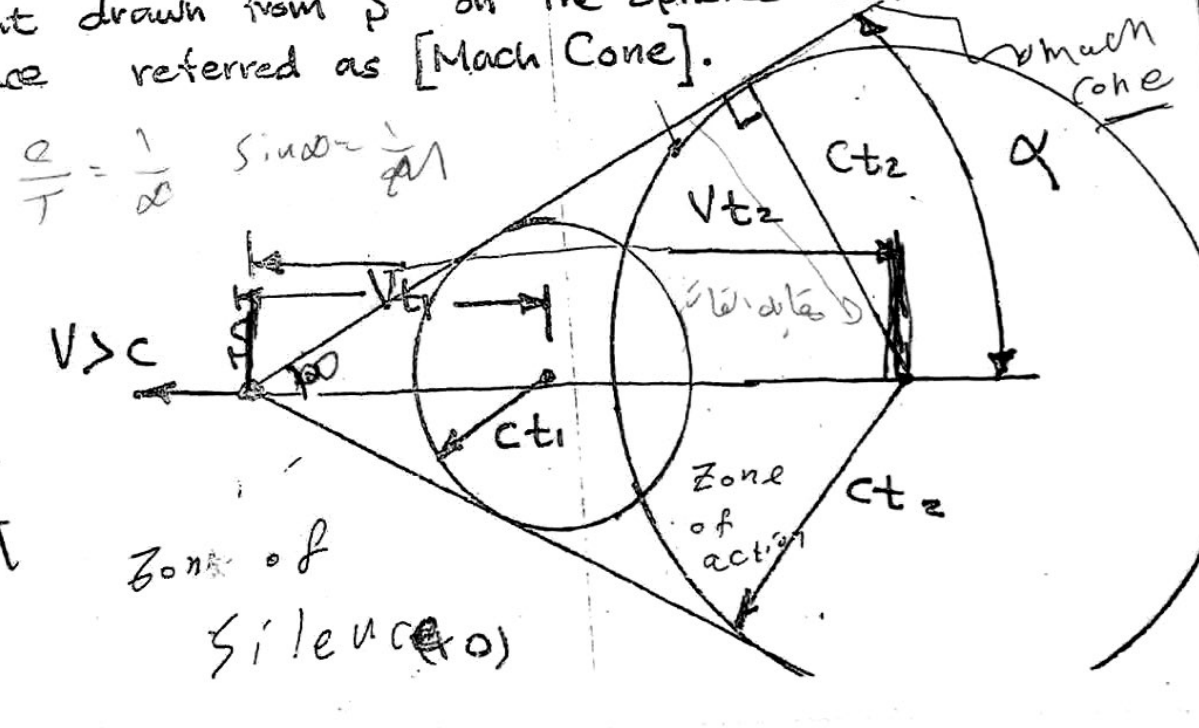


$V < c$

(3) When the point source travels with the same wave velocity ($V = c$), the flow is sonic (M). The wave fronts always exist downstream of the point source. The zone to the left is called zone of action and to the right of S is called zone of action.



(4) When the point source travels with velocity V higher than c, ($V > c$) the flow is supersonic ($M > 1$). The point source S is always ahead of the wave fron. Tangent drawn from S on the spheres define a conical surface referred as [Mach Cone].



$\sin\alpha = \frac{1}{M}$

The semi-angle α is called and known as Mach An[gle]

From the supersonic spherical waves figure,

$$\alpha = \sin^{-1}\frac{c t_2}{V t_2} = \sin^{-1}\frac{c t_1}{V t_1} = \sin^{-1}\frac{c}{V} = \sin^{-1}\frac{1}{V/c}$$

so; $$\boxed{\alpha = \sin^{-1}\frac{1}{M}} \qquad (2.6)$$

Solved Examples

$M = \frac{V}{c}$ $V = M\sqrt{KRT}$

$V^2 = KRT$

EX(1):

An air-craft is capable of flying at a maximum Mach number of 0.91 at sea-level. Find the maximum velocity at which this aircraft can fly at sea-level the air temperature is (a) 5°C and (b) 45°C.

Solution

$M = 0.91$ $V = ?$ at $t = 5°$ and $45°$

Since $\left(M_{max} = \frac{V_{max}}{c}\right)$

it follows that $V_{max} = M_{max}\, c = M_{max}\sqrt{kRT_{se}}$

(a) when T at sea-level $= 5°C$

$T = 5 + 273 = 278\ K$, $R = 287\ J/kg\,K$, $k = 1.4$

$V_{max} = 0.91\sqrt{1.4 \times 287 \times 278} = 304\ m/s$

$t = T$

(b) When T at sea-level $= 45°C$, $T = 45 + 273 = 318$

$V_{max} = 0.91\sqrt{1.4 \times 287 \times 318} = 325\ m/s$

$W = \pi D N \times R$

EX(2) An aircraft is driven by a propellers with diameter of $\boxed{4\ m}$. At what speed (engine speed) will the tips of the propeller reach sonic speed if the air temperature is 15°C?

$M = 1$

$V^2 = KRT$

Solution For sonic case $M = 1$ and $V = c$

$D = 4\,m$. $c = \sqrt{\ }$ $\therefore V = \sqrt{kRT}$

$W = \frac{\pi D N \times 4}{60} = \sqrt{KRT}$

(11)

The semi-angle α is called and known as Mach Angle

From the supersonic spherical waves figure,

$$\alpha = \sin^{-1}\frac{ct_2}{Vt_2} = \sin^{-1}\frac{ct_1}{Vt_1} = \sin^{-1}\frac{c}{V} = \sin^{-1}\frac{1}{V/c}$$

so; $$\left[\alpha = \sin^{-1}\frac{1}{M}\right] \quad (2.6)$$

Solved Examples $M = \frac{V}{C}$ $V = M\sqrt{KRT}$

$V^2 = KRT$

EX(1):

An air-craft is capable of flying at a maximum Mach number of 0.91 at sea-level. Find the maximum velocity at which this aircraft can fly at sea-level the air temperature is (a) $5^\circ C$ and (b) $45^\circ C$.

Solution $M = 0.91$ $V = ?$ at $t = 5^\circ$ and 45°

Since $M_{max} = \frac{V_{max}}{c}$

it follows that $V_{max} = M_{max}\,C = M_{max}\sqrt{kRT_{sea}}$

(a) when T at sea-level $= 5^\circ C$

$T = 5 + 273 = 278\ K$, $R = 287\ J/kg\,K$, $k = 1.4$

$V_{max} = 0.91\sqrt{1.4 \times 287 \times 278} = 304\ m/s$

$^\circ k = T$

(b) When T at sea-level $= 45^\circ C$, $T = 45 + 273 = 318$

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$M = 1$ $V^2 = KRT$

Solution For sonic case $M = 1$ and $V = c$

$D = 4m$. $c = \text{sound}$ $V = \sqrt{kRT}$

$w.\frac{\pi D N}{60} \times 4 = \sqrt{KRT}$ (11)

$w = \frac{2\pi N}{60} \cdot R$

$w \cdot R$

$V = \pi N D$ and $N = \frac{V}{\pi D} = \frac{\sqrt{1.4 \times 287 \times (15+273)}}{\pi \times 4}$

$V = 27.06$ rps

$V = 27.06 \times 60 = 1623$ rpm

$w = \frac{2\pi N}{60}$ $V = \frac{2\pi R N}{60} = \frac{D}{2}$

EX(3) مثال

The cruising speed of Boeing 747 (F16.) is 978 km/h at an altitude of 9150 m and that of Concorde is 2340 km/h at an altitude of 16000 m. Find the Mach number of the aircraft at the cruising condition.. Take: $T = 288.16 - (0.0065 H)$

Solution

Boeing 747: $V = 978$ km/h $= 271.7$ m/s

$T = 288.16 - 0.0065 \times 9150 = 228.7$ K

$c = \sqrt{1.4 \times 287 \times 228.7} = 303 \frac{m}{s}$

$\therefore M = \frac{V}{c} = \frac{271.7}{303} = 0.897$

Concorde: $V = 2340$ km/h $= 650$ m/s

$T = 288.16 - 0.0065 \times 16000$

$T = 216.66$ K

$c = \sqrt{1.4 \times 287 \times 216.66} = 294.9$ m/s

$M = \frac{V}{c} = \frac{650}{294.9} = 2.204$

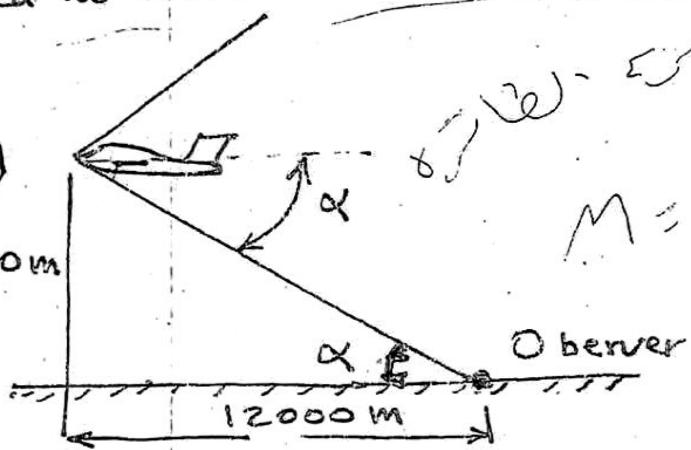
Note: Since $M < 1$ for Boeing aircraft which is consider as a subsonic plane, which the concorde is a supersonic plane since $M > 1$.

Cone

EX (4)

An observer on the ground find that an airp[lane] fly horizontally at an altitude of 5000 m ha[s] traveled 12 km from the overhead position bef[ore] the sound of the airplane is first heard. Estima[te] the speed at which the airplane is flying.

$\sin^{-1}\alpha =$



$M = 2.2$

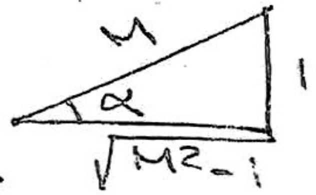
$M = \frac{1}{\sin\alpha}$

Solution The temperature at mean altitude of 25[00] m, $T = 288.16 - 0.0065 \times 2500 = 271.9\,K$

the temperature evaluated at mean altitude since the actual Mach waves ~~at~~ from the aircraft are curved, so, the average sound velocity at average temperature between 0 and 5000 m being used to describe the Mach number.

$c = \sqrt{1.4 \times 287 \times 271.9} = 330.6\ m/s$

$\tan\alpha = \frac{5000}{12000} = 0.417$



$\sin\alpha = \frac{1}{M}$, so $\tan\alpha = \frac{1}{\sqrt{M^2-1}}$

$\sin\alpha = \frac{V}{c}$ $0.417 = \frac{1}{\sqrt{M^2-1}}$, $M = \sqrt{\left(\frac{1}{.417}\right)^2 + 1} = 2.6$

$\therefore$ Velocity of aircraft $= V = M \times c$

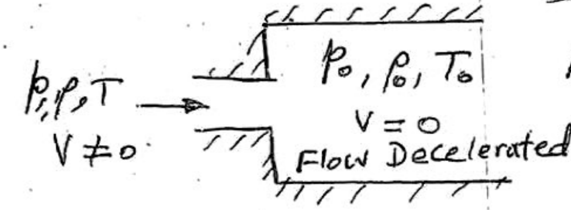
$V = 2.6 \times 330.6 = 859\ m/s$

~~$\frac{5000}{12000}$~~ ~~$\frac{5000}{1200}$~~

$\frac{5000}{13000}$ tan α

Reference Velocities and Conditions V.I.P

(1) Stagnation State When the flow deceleration to zero velocity, so the pressure and temperature increase, on other hand: the gas flow accelerated to non zero velocity. Throughout the course the stagnation state is denoted by subscript 'o', i.e, T_o, P_o, ρ_o and h_o, $V_o = 0$, $M = 0$

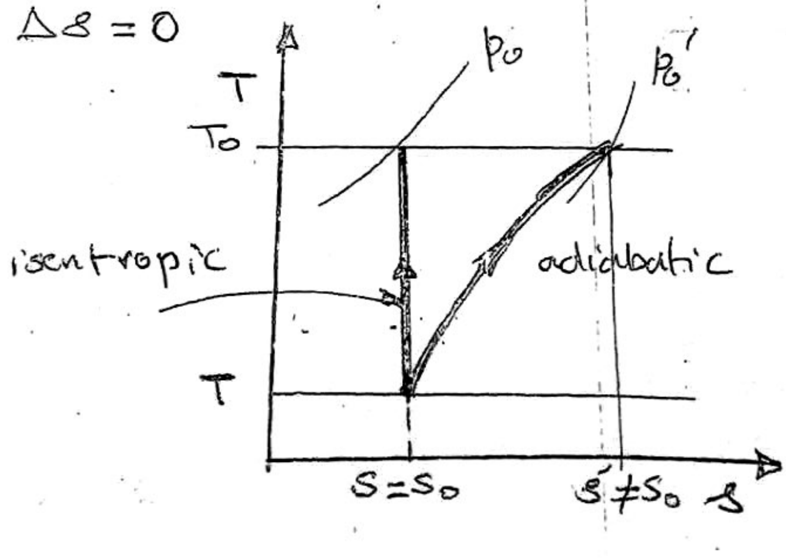


All tanks and reserviors are considered as stagnation states.

(2) Critical state or Sonic Condition,
When the gas velocity is equal to sound-Velocity so, the Mach number is unity, this condition is denoted by superscript (*), $\{V = C$ and $M = 1\}$

Sonic $M=1$

(3) Isentropic Process
~~The expansion~~ and compression of gas is isentropic = reversible + adiabatic
$\Delta s = 0$



Recall equation (7.1) page 4, that is,

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = \text{constant} \qquad (27)$$

apply this equation to the process in the previous fig
$h_1 = h_0$ = stagnation enthalpy or total enthalpy which is the maximum enthalpy in the process

$V_1 = 0$
$h_2 = h$, $V_2 = V$) so

$$h_0 = h + \frac{V^2}{2} \qquad (28)$$

$h_0 = C_p T_0$, $h = C_p T$ substitute in (28)

$$T_0 = T + \frac{V^2}{2C_p} \qquad (29)$$

C_0

(4) Maximum Fluid Velocity V_{max}.

The maximum velocity achieved by fluid when it is accelerated to absolute zero temperature ($-273\,^\circ C$), means that $h = 0$, $T = 0$
recall equation (28) $\therefore$ $V_{max} = \sqrt{2h_0}$ (30)

or $V_{max} = \sqrt{2C_pT_0}$, $C_p = \frac{k}{k-1} R$ From equ.(1

$$\therefore V_{max} = \sqrt{\frac{2k}{k-1} R T_0} \qquad (31)$$

(5) Sound Velocity of Stagnation State (C_0)

C_0 is the sound velocity evaluated at stagnation T_0,

$$C_0 = \sqrt{kRT_0} \qquad (32)$$

From (31) $V_{max} = \sqrt{\frac{2}{k-1}} \sqrt{kRT_0} = \sqrt{\frac{2}{k-1}}\, C_0$

So, $V_{max}/C_0 = \sqrt{2/(k-1)}$ —— (33)

(6) <u>Critical Velocity of Sound</u> c^*

which is <u>evaluated</u> at T^*,

$$c^* = \sqrt{kRT^*} = V^* \quad , \quad M = 1 \qquad \text{——— (34)}$$

applying energy equation between stagnation state and critical <u>state</u>, $h_o = h^* + \frac{c^{*2}}{2}$ or $T_o = T^* + \frac{c^{*2}}{2C_p}$

$$\therefore \; c^* = \sqrt{2C_p(T_o - T^*)} \qquad (35)$$

$$\text{or} \quad c^* = \sqrt{\frac{2k}{k-1} R(T_o - T^*)} = V^* \qquad (36)$$

Squaring (36) $V^{*2} = \frac{2}{k-1}\left[kRT_o - kRT^*\right] = \frac{2}{k-1}\left[c_o^2 - V^*\right]$

$$\frac{2}{k-1} V^{*2} + V^{*2} = \frac{2}{k-1} c_o^2$$

$$\frac{k+1}{k-1} V^{*2} = \frac{2}{k-1} c_o^2 \quad \text{or} \quad \frac{V^*}{c_o} = \sqrt{\frac{2}{k+1}} \qquad \text{——— (37)}$$

Dividing equation (33) and (37)

$$\frac{V_{max}}{c^*} = \sqrt{\frac{k+1}{k-1}} \qquad \text{——— (38)}$$

(7) M^* or The Mach number referred to critical condition

$$M^* = \frac{V}{c^*} = \frac{V}{V^*} \qquad \text{——— (39)}$$

Multiply (39) by c and divide by c,

$$M^* = \frac{V}{c} \cdot \frac{c}{c^*} = \frac{c}{c^*} M \qquad \text{——— (40)}$$

<u>Bernoulli Equation</u> $M \ngtr 0.3$ $M < 0.3$

This equation can be used only for <u>incompressible</u> fluid. From equation (28)

$$h_o = h + \frac{V^2}{2} = \text{constant} \quad , \text{ differentiate it,}$$

$$d(h_o) = d(\text{const.}) = 0 = dh + \boxed{V\,dV} \qquad \text{——— (41)}$$

From (15) and isentropic flow, $ds = 0$

$$T \cdot ds = dh - \frac{dP}{d\rho} \qquad T \cdot ds = P \Delta ...$$

$h = PV + u$

$dh = \rho dv + v\, dp + du$

$0 = dh - \frac{dp}{\rho}$ (42)

$T\,ds = dh - \frac{1}{\rho}\, dp$

the flow is incompressible, ρ = constant

Substitute dh in (42) in (41)

$$\boxed{\frac{dp}{\rho} + V\,dV = 0} \quad (41a) \text{ integrating}$$

$$\frac{1}{\rho}\int dp + \int V\,dV = \frac{p}{\rho} + \frac{V^2}{2} = \text{constant} \quad (43)$$

$\rho = \rho_o$ (incompressible), $V_o = 0$, $P = P_o$

$$\boxed{\therefore\; P_o = P + \tfrac{1}{2}\rho V^2} \text{ or } \frac{P}{\rho} + \frac{V^2}{2} = \text{const} \quad (44)$$

Gas isentropic

$ds = 0$

The energy equation is also used to make another form of Bernoulli equation

$h_o = h + \frac{V^2}{2}$,

$h_o = C_p T_o = \frac{k}{k-1} R T_o$, From equation of state

$R T_o = \frac{P_o}{\rho_o}$ and $RT = \frac{P}{\rho}$

So, $h_o = \frac{k}{k-1} \frac{P_o}{\rho_o}$ and similarly $h = \frac{k}{k-1} \frac{P}{\rho}$

$\rho R T = P$

and the energy equation is then

$$\frac{k}{k-1}\frac{P}{\rho} + \frac{V^2}{2} = \frac{k}{k-1}\frac{P_o}{\rho_o} \quad (45)$$

$$\boxed{\frac{k}{k-1}\frac{P_o}{\rho_o} = \frac{k}{k-1}\frac{P}{\rho} + \frac{V^2}{2}} \qquad \frac{P_o}{\rho} = \frac{P}{\rho} + \frac{V^2}{2}$$

compressible

$\rho = \rho_o$ (incomp)

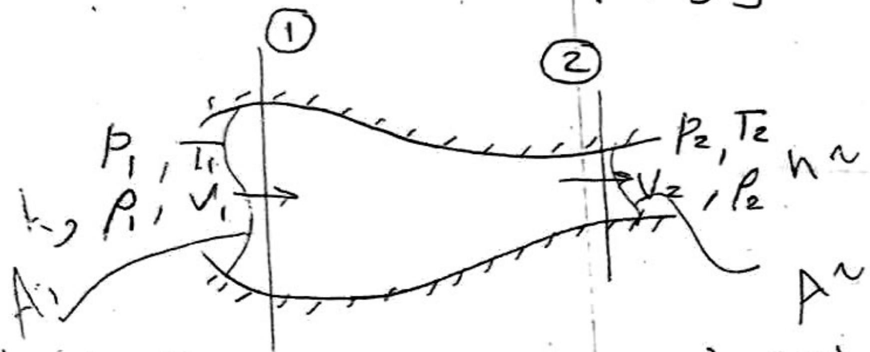
$\rho_o \neq \rho$

(17)

for

One-Dimensional Compressible Flow

Assuming isentropic Flow, between section 1 and 2 in an adiabatic conduit of varying area.



Applying the energy equation in (8)

$$T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p},$$

Taking the second term $\frac{V^2}{2c_p}$ multiplying and dividing by T

$$\frac{V^2}{2c_p}\frac{T}{T} = \frac{k-1}{2k}\frac{V^2}{RT}$$

$$= \frac{k-1}{2}\frac{V^2}{kRT}T, \quad kRT = c^2$$

$kRT \to c^2$

$$\therefore \frac{V^2}{2c_p} = \frac{k-1}{2}\frac{V^2}{c^2}T = \frac{k-1}{2}M^2 T$$

$$\frac{V_1^2}{2c_p} = \frac{k-1}{2}M_1^2 T_1, \quad \frac{V_2^2}{2c_p} = \frac{k-1}{2}M_2^2 T_2$$

the new form of energy equation is

$$T_1 + \frac{k-1}{2}M_1^2 T_1 = T_2 + \frac{k-1}{2}M_2^2 T_2$$

$$T_1\left(1 + \frac{k-1}{2}M_1^2\right) = T_2\left(1+\frac{k-1}{2}M_2^2\right)$$

$$\boxed{\frac{T_2}{T_1} = \frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2}} \quad (46)$$

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}}$$

For isentropic relation in (18) and (19)

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{k/k-1}$$

$$\therefore \frac{p_2}{p_1} = \left[\frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2}\right]^{\frac{k}{k-1}} \quad (47)$$

(18)

$1/k-1$

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}}$$

$$\therefore \frac{\rho_2}{\rho_1} = \left[\frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2}\right]^{\frac{1}{k-1}} \quad (48)$$

Now suppose we want to use equations 46, 47 and 48 in terms of reference state, like the stagnation condition.

let $T_2 = T$ any state within the flow
$P_2 = P$, $M_2 = M$
$\rho_2 = \rho$ and $P_1 = P_0$, thus $M_1 = 0$
$T_1 = T_0$, $\rho_1 = \rho_0$ and

equation (46) becomes as well as equations 47 & 48

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}M^2\right)^{-1} \quad (49)$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}M^2\right)^{-\frac{k}{k-1}} \quad (50)$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}M^2\right)^{-\frac{1}{k-1}} \quad (51)$$

For the above equation one can form relations between two reference states, critical state ($M = 1$) and stagnation state ($M = 0$)

$T = T^*$ $P = P^*$, $\rho = \rho^*$, $M = 1$

$$\frac{T_0}{T_{1\,or\,2}} = 1+\frac{k-1}{2}M^2 \quad (52)$$

$$\frac{P_0}{P_{1\,or\,2}} = \left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}} \quad (53)$$

$$\frac{\rho_0}{\rho_{1\,or\,2}} = \left(1+\frac{k-1}{2}M^2\right)^{\frac{1}{k-1}} \quad (54)$$

From (52), $\frac{T_o}{T^*} = 1 + \frac{k-1}{2}(1)^2 = \frac{k+1}{2}$

$\therefore \frac{T^*}{T_o} = \frac{2}{k+1}$ (55), for air $k = 1.4$

$$\boxed{\frac{T^*}{T_o} = 0.8333} \quad (56)$$

From (53) $\frac{p_o}{p^*} = \left[1 + \frac{k-1}{2}(1)^2\right]^{\frac{1.4}{1.4-1}} = \left(\frac{k+1}{2}\right)^{-3.5}$ (56)

$$\therefore \frac{p^*}{p_o} = (0.8333)^{3.5} = 0.528, \; k = 1.4 \quad (57)$$

Similarly $\frac{\rho^*}{\rho_o} = 0.634$, $k = 1.4$ (58)

A/A^*, ρ/ρ^*

Examples

EX(1) Show that the energy equation has the form

$$h_o = \frac{c^2}{k-1} + \frac{V^2}{2}$$

ما معنى

Solution $h_o = h + \frac{V^2}{2}$

$h = C_p T = \frac{k}{k-1} RT = \frac{c^2}{k-1}$, since $c = \sqrt{kRT}$

$$\therefore h_o = \frac{c^2}{k-1} + \frac{V^2}{2}$$

$C^* = V^*$

EX(2) An air jet 300 K has sonic velocity, determine the following: $M = 1$

1. velocity of sound
2. velocity of sound at stagnation conditions c_{T_o}, T
3. Maximum jet velocity →
4. Stagnation enthalpy

عندما يقول هذه العبارة معناه $T = 0$ ساكنة

أعلى سرعة

$T_o = \frac{V_e}{2C_p}$

$\frac{v_{max}}{2}$ $\quad$ $h_o = h + \frac{v^2}{2}$

Solution For sonic velocity $M = 1$ and $c = V$

1. $c = \sqrt{kRT} = \sqrt{1.4 \times 287 \times 300} = 347.18$ m/s
2. $c_o = \sqrt{kRT_o}$, equation (32)

$\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2$, $M = 1$

$\therefore T_o / T = 1 + \frac{(1.4-1)}{2} = 1.2$

$T_o = 1.2 \times 300 = 360$ K

$c_o = \sqrt{1.4 \times 287 \times 360} = 380.32$ m/s

3. From equation (33)

$V_{max} / c_o = \sqrt{\frac{2}{k-1}}$ $\quad$ $h = c_p T_o$

$V_{max} = c_o \sqrt{\frac{2}{k-1}} = 380.32 \sqrt{\frac{2}{1.4-1}} = 850.42$ m/s

4. From energy equation for $V = V_{max}$, $h = 0$

$h_o = \frac{1}{2} V_{max}^2 = \frac{1}{2} (850.42)^2 = 361.6 \times 10^3$ J/kg

$h_o = h + \frac{v^2}{2}$ or, $h_o = c_p T_o = \frac{k}{k-1} R T_o = \frac{1.4}{1.4-1} \times 287 \times 360$

$\frac{k}{k-1} R$ $\quad$ $h_o = 361.6 \times 10^3$ J/kg

$h_o = c_p T_o$ EX(3)

A jet of gas at 500 K has a Mach number of 1.2 determine;

1. Local sound velocity $\quad$ $c = \sqrt{kRT}$
2. Stagnation sound velocity $\quad$ $c_o = \sqrt{kRT_o}$
3. Static enthalpy $\quad$ 1.4
4. Stagnation enthalpy $\quad$ 287
5. Maximum jet velocity , take $k = 1.3$, $R = 469$

Solution

1. $c = \sqrt{kRT} = \sqrt{1.3 \times 469 \times 500} = 552.13$ m/s
2. $\frac{T_o}{T} = 1 + \frac{k-1}{2} M^2 = 1 + \frac{1.3-1}{2} (1.2)^2 = 1.216$

$T_o = 1.216 \times 500 = 608$ K

$c_o = \sqrt{kRT_o} = \sqrt{1.3 \times 469 \times 608} = 608.85$ m/s

3. $h = C_p T = \frac{k}{k-1} RT = \frac{1.3}{1.3-1} \times 469 \times 500 = 10.16 \times 10^5$

4. $h_o = C_p T_o = \frac{1.3}{1.3-1} \times 469 \times 608 = 12.3 \times 10^5 \ \frac{J}{kg}$

$h_o = 12.3566 \times 10^5$ J/kg

5. Maximum jet velocity $V_{max} \longrightarrow T = 0$

$h_o = \frac{1}{2} V_{max}^2 \quad (h = 0 \text{ for } T = 0)$

$V_{max} = \sqrt{2 h_o} = \sqrt{2 \times 12.3566 \times 10^5} = 1572$ m/s

$V_{max} \to T = 0$ (k)
$M = \infty$

or $V_{max} = \sqrt{\frac{2}{k-1}} * C_o = \sqrt{\frac{2}{1.3-1}} \times 608.85$

$V_{max} = 1572$ m/s

EX(4) Air enters a straight duct at 250 kPa and 30°C. The inlet Mach number is 1.5 and exit Mach number is 2.4, assuming isentropic flow take $k = 1.4$ and $R = 287$ J/kg K, Determine:

1. Stagnation temperature
2. Exit local temperature and velocity
3. Exit pressure
4. Mass flow rate per unit area.

Solution

$\frac{T_o}{T_1} = 1 + \frac{k-1}{2} M_1^2 = 1 + \frac{1.4-1}{2} (1.5)^2 = 1.45$

$T_o = 1.45 \times (30+273) = 439.35$ K $\quad (T_{o1} = T_{o2})$

$T_1 = 30 + 273 = 303$ K

2. $\frac{T_o}{T_2} = 1 + \frac{k-1}{2} M_2^2 = 1 + \frac{1.4-1}{2} (2.4)^2 = 2.152$

$\therefore T_2 = \frac{T_o}{2.152} = \frac{439.35}{2.152} = 204.15$ K

$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{k/k-1} = \left(\frac{204.15}{303}\right)^{\frac{1.4}{1.4-1}} = 0.251$??

??

3. $\therefore p_2 = 0.251 \times 250 = 62.764$ kPa exit pressu[re]

exit velocity, $c_2 = \sqrt{kRT_2} = \sqrt{1.4 \times 287 \times 204.15} = 286.$[4] m/s

$M_2 = \frac{V_2}{c_2} \therefore V_2 = M_2 c_2 = 2.4 \times 286.4 = 687.36$ m/s

4. Mass flow rate $\dot{m} = A_1 \rho_1 V_1 = A_2 \rho_2 V_2$

$A_1 = A_2$ straight duct

$$\frac{\dot{m}}{A_2} = \rho_2 V_2 = \frac{p_2 V_2}{RT_2}$$

$$\frac{\dot{m}}{A_1} = \rho_1 V_1 = \frac{p_1}{RT_1} M_1 c_1 = \frac{p_1}{RT_1} M_1 \sqrt{kRT_1}$$

$$= \sqrt{\frac{k}{RT_1}}\, p_1 M_1 = \sqrt{\frac{1.4}{287 \times 303}} \times 250 \times 10^3 \times 1.5$$

$$\frac{\dot{m}}{A_1} = 1502.1 \ \frac{kg}{s\, m^2}$$

Homework Problem

The pressure, temperature and velocity of air at entry of flow passage are 300 kPa, 280 K and 140 m/s. The pressure, temperature and velocity at exit are 200 kPa, 260 K and 250 m/s. The area of the cross section at entry is 600 cm^2, determine for adiabatic flow

1. stagnation temperature
2. Maximum velocity
3. Mass flow rate
4. Exit cross section area

$k = 1.4$, $R = 287$ J/kgK

Answers

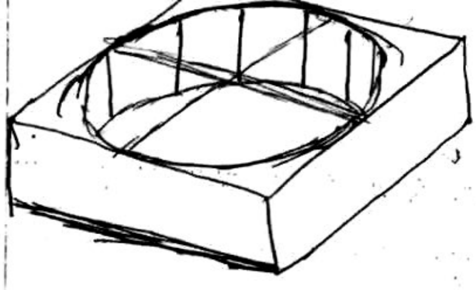
$T_o = 289.7$ K

$V_{max} = 762.9$ m/s

$\dot{m} = 31.36$ kg/s

$A_2 = 0.0468$ m^2

مهم



Area Ratio A/A^*

From continuity equation;

$$\rho A V = \rho^* A^* V^* \quad (M=1,\ V^* = c^*) \qquad (59)$$

so;
$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V} \qquad (60)$$

but
$$\frac{V^*}{V} = \frac{c^*}{V} = \frac{\sqrt{kRT^*}}{M\sqrt{kRT}} = \frac{1}{M}\sqrt{\frac{T^*}{T}} = \frac{1}{M}\sqrt{\frac{T^*}{T_0}}\sqrt{\frac{T_0}{T}}$$
$$= \frac{1}{M}\sqrt{\frac{T^*}{T_0}\frac{T_0}{T}} \qquad (61)$$

From (55) $\frac{T^*}{T_0} = \frac{2}{k+1}$, From (52) $\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$

$$\therefore \frac{V^*}{V} = \frac{1}{M}\sqrt{\frac{2}{k+1}\left(1+\frac{k-1}{2}M^2\right)} = \frac{1}{M}\sqrt{\frac{2}{k+1} + \frac{k-1}{k+1}M^2} \qquad (62)$$

From (48) $\frac{\rho_2}{\rho_1} = \left(\frac{1+\frac{k-1}{2}M_1^2}{1+\frac{k-1}{2}M_2^2}\right)^{1/k-1}$

let $\rho_2 = \rho^*$, $M_2 = 1$, $\rho_1 = \rho$, $M_1 = M$

$$\therefore \frac{\rho^*}{\rho} = \left[\left(1+\frac{k-1}{2}M^2\right)/\left(1+\frac{k-1}{2}\right)\right]^{1/k-1}$$

$$\frac{\rho^*}{\rho} = \left[\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{\frac{1}{k-1}} = \left[\frac{2}{k+1} + \frac{k-1}{k+1}M^2\right]^{\frac{1}{k-1}} \quad (62$$

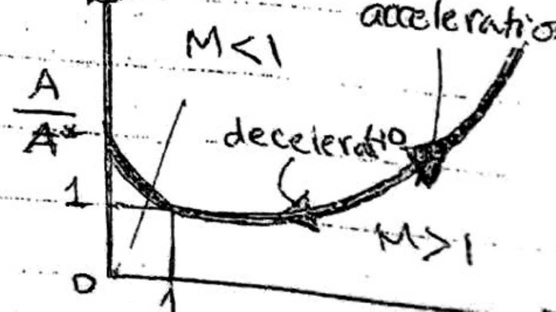
substitude (61) & (62) into (60)

$$\frac{A}{A^*} = \left[\frac{2}{k+1} + \frac{k-1}{k+1}M^2\right]^{\frac{1}{k-1}} \times \frac{1}{M}\left[\frac{2}{k+1} + \frac{k-1}{k+1}M^2\right]^{1/2}$$

$$\left(\frac{1}{k-1}\right) + \frac{1}{2} = \frac{2+k-1}{2(k-1)} = \frac{k+1}{2(k-1)}$$

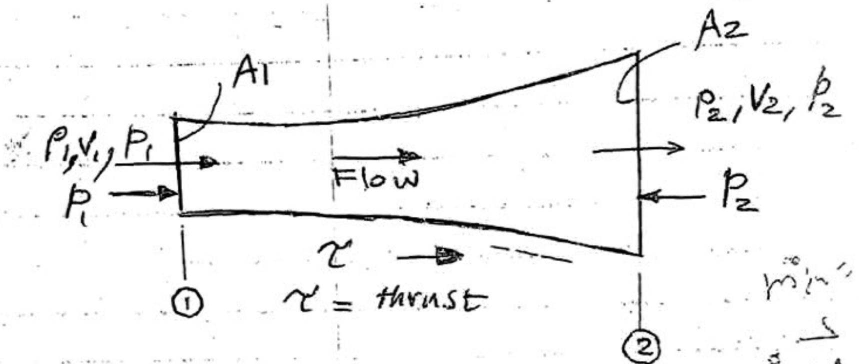
$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{k+1} + \frac{k-1}{k+1}M^2\right]^{\frac{k+1}{2(k-1)}} \qquad (63)$$

Plot of equation (63) is shown in the right hand graph.



Impulse Function F

Consider a symmetrical straight duct in the ... below. The thrust or wall forces experienced by the d... in direction shown result of change in pressure and M... number between the cross-sections 1 and 2.



τ = thrust

$\dot m_{in} = \dot m_{out}$

Applying Newton's law of Motion: $\sum \vec{F} = \dot{m}\vec{V}$

$$P_1A_1 + \tau - P_2A_2 = \dot m V_2 - \dot m V_1 \quad , \quad \dot m = \rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

$$P_1A_1 + \tau - P_2A_2 = \rho_2 A_2 V_2^2 - \rho_1 A_1 V_1^2$$

$$\tau = (P_2A_2 + \rho_2A_2V_2^2) - (P_1A_1 + \rho_1A_1V_1^2) \qquad (64)$$

$$\rho V^2 = \rho M^2 c^2 = \rho M^2 kRT = kM^2(\rho RT) = kM^2 P \qquad (6\ldots)$$

$$\therefore \tau = (P_2A_2 + P_2A_2kM_2^2) - (P_1A_1 + P_1A_1kM_1^2)$$

$$\tau = P_2A_2(1+kM_2^2) - P_1A_1(1+kM_1^2) \qquad (66)$$

$\tau = F_2 - F_1$, F is the impulse function

$$F = PA(1+kM^2) \quad \text{in general} \qquad (67)$$

$$F^* = P^*A^*(1+k) \quad , \quad M = 1 \qquad (68)$$

and, $$\frac{F}{F^*} = \frac{P}{P^*}\,\frac{A}{A^*}\,\frac{1+kM^2}{1+k} \qquad (69)$$

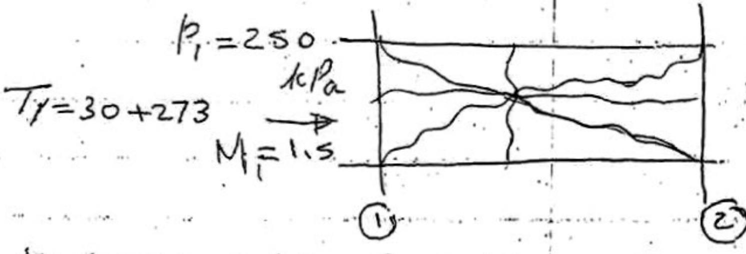
substitute $\frac{P}{P^*}$ From equation (47) assuming $P_2 = P$ and $P_1 = P^*$ and $\frac{A}{A^*}$ From (63) into equation (69) to end with:

$$\boxed{\frac{F}{F^*} = \frac{1+kM^2}{M\sqrt{2(1+k)\left(1+\frac{k-1}{2}M^2\right)}}} \qquad (70)$$

Gas Tables

All equations derived $M, M^*, \frac{T}{T_0}, \frac{p}{p_0}, \frac{A}{A^*}, \frac{F}{F^*}$

have been tabilated for different values of M, to make the solutions of gas easier in engineering design of nozzles, diffusers, jet engines and turbines rather than using equations. Keep in mind that all these tables are evaluated for air that is $\boxed{K=1.4}$.
To use the gas table, recall the data from example(4) page (22)



Enter isentropic table for $M_1 = 1.5$ to get:

$$\frac{P_1}{P_0} = 0.2724, \quad \frac{T_1}{T_0} = 0.68965$$

1. $$T_0 = \frac{T_1}{0.68965} = \frac{303}{0.68965} = 439.35 \text{ K}$$

OR from $M_2 = 2.4$, $\frac{p_2}{p_0} = 0.068399$, $\frac{T_2}{T_0} = 0.46468$

2. $$T_2 = \frac{T_2/T_0}{T_1/T_0} \times T_1 = \frac{0.46468}{0.68965} \times 303 = 204.158 \text{ K}$$

$$V_2 = M_2 C_2 = 2.4\sqrt{1.4 \times 287 \times 204.158} = 286.4 \ \frac{m}{s}$$

$$P_2 = \frac{P_2/P_0}{P_1/P_0} \times P_1 = \frac{0.068399}{0.2724} \times 250 = 62.77 \text{ kPa}$$

$$\mathring{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$\frac{\mathring{m}}{A_1} = \sqrt{\frac{k}{RT_1}}\, P_1 M_1 = \sqrt{\frac{1.4}{287 \times 303}} \times 250 \times 10^3 \times 1.5 = 1502 \ \frac{kg}{s}$$